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A symmetry classification of superfluid ^{3}He phases

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Abstract. A symmetry classification of the superfluid ^{3}He phases is presented that is based on group representation theory. Using this classification scheme, an analysis of the superfluid cores of ^{3}He-B vortices is given.

1. Introduction

A characteristic feature of superfluids is the broken $\mathrm{U}(1)^\Phi$ gauge invariance. In the case of liquid ^{3}He the dynamics has an additional $\mathrm{SO}(3)^S \times \mathrm{SO}(3)^L$ symmetry, where S refers to spin rotations and L to ordinary space rotations, which is also broken in the superfluid state. (We consider the S- and L-rotations as independent symmetries, because we neglect the small dipole spin–orbit interaction.) The description of superfluid ^{3}He involves an order parameter $A_{\alpha i}$, $\alpha, i = 1, 2, 3$, with α the spin index and i the orbital index [1]. This (complex) object transforms non-trivially under $\mathrm{U}(1)^\Phi$ and as the $(\underline{3}^S, \underline{3}^L)$ representation under $\mathrm{SO}(3)^S \times \mathrm{SO}(3)^L$, i.e. it transforms as a vector under both factors [1]. For our purposes it is more convenient to work in a spherical basis where the $(\underline{3}^S, \underline{3}^L)$ has (complex) entries $a_{m_S m_L}$, with $m_{S,L}$ the weights ($m_{S,L} = 0, \pm 1$). The different superfluid phases are now characterised by different constant values of the order parameter $a_{m_S m_L}$. Alternatively, we may characterise the phase by the residual symmetry $H \subset G$ which leaves the corresponding value of $a_{m_S m_L}$ invariant and where G is the full symmetry [2],

$$G = \mathrm{SO}(3)^S \times \mathrm{SO}(3)^L \times \mathrm{U}(1)^\Phi. \tag{1.1}$$

The transformation properties of the relevant order parameter follow from the assumed pairing mechanism in the underlying microscopic theory. All phases that can possibly be realised, as well as many of the physical properties of these phases, are a direct consequence of the representation content of the order parameter.

There are basically two approaches for determining the possible phases. The first consists of explicitly minimising the energy functional of the Ginzburg–Landau theory. An extensive analysis of this kind has been given in [3]. A second approach is based on group theory, like the one followed by Bruder and Vollhardt [2]. They systematically classified all subgroups H of the full symmetry group G, and subsequently determine whether a breaking of G is possible with a particular choice of the order parameter.

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In this paper we again rely on group theory, but in contrast with the work in [2], we directly start from the representation theory of the $(\underline{3}^S, \underline{3}^L)$ representation, which in fact leads to equivalent results in a much more direct and simple way. Section 2 deals with the continuous residual symmetries and § 3 with the discrete ones. As this paper basically does not contain any new results we think its interest resides in the simple method by which we were able to derive the known results.

The method we employ is also very useful in determining the symmetry-breaking patterns that occur in topological defects, and in studying the superfluid ^{3}He phases in an external magnetic field of increasing strength. As an illustration, we discuss the superfluid cores of ^{3}He-B vortices in § 4. In recent years vortex phenomena in superfluid ^{3}He have received considerable attention in the literature (for a review see [4])—especially since 1982 when some surprising NMR data on rotating superfluid ^{3}He-B were reported [5].

2. Continuous symmetries

In this section we enumerate all the superfluid ^{3}He phases with a continuous residual symmetry. The largest subgroup to which the symmetry group (1.1) can be broken down is the diagonal subgroup $SO(3)^J$ of $SO(3)^S \times SO(3)^L$, where $\boldsymbol{J} = \boldsymbol{S} + \boldsymbol{L}$. The branching rule is

$$G \supset SO(3)^J \qquad (\underline{3}^S, \underline{3}^L) = \underline{5}^J + \underline{3}^J + \underline{1}^J. \tag{2.1}$$

The last term in (2.1) indicates that the order parameter of this so-called B phase is a singlet under $SO(3)^J$, i.e. the order parameter is indeed invariant under the residual symmetry transformations $SO(3)^J$. The diagonal subgroup is the only SO(3) group to which the symmetry group of liquid ^{3}He can be broken down. This may be understood as follows. Let $rS_\mu + sL_\mu$, $\mu = 1, 2, 3$, be the generators of $SO(3)^H$, where $\boldsymbol{S}$ and $\boldsymbol{L}$ are the generators of $SO(3)^{S,L}$, respectively. The generators of the residual symmetry group should satisfy the SO(3) algebra; hence

$$[rS_\mu + sL_\mu, rS_\lambda + sL_\lambda] = \mathrm{i}\varepsilon_{\mu\lambda\nu}(rS_\nu + sL_\nu) \tag{2.2}$$

or

$$r^2 S_\nu + s^2 L_\nu = rS_\nu + sL_\nu. \tag{2.3}$$

The solutions of this equation are $r = s = 1$, which corresponds to the diagonal subgroup $SO(3)^J$, $r = 0$, $s = 1$ and $r = 1$, $s = 0$. The last two solutions would imply that either $SO(3)^S$ or $SO(3)^L$ remains unbroken, which is impossible for a superfluid in the $(\underline{3}^S, \underline{3}^L)$ representation.

To determine all possible continuous Abelian subgroups it suffices to restrict ourselves to rotations around the third axes in spin and orbit space, denoted by $\Omega_3^{S(L)}$. In our basis the tensor component $a_{m_S m_L}$ transforms with a phase factor only. If we consider a transformation $\Omega_3^S(\theta_S)\Omega_3^L(\theta_L)\Omega^\Phi(\varphi)$, where Ω^Φ refers to a $U(1)^\Phi$ gauge transformation, then

$$a_{m_S m_L} \to a'_{m_S m_L} = \exp[\mathrm{i}(\theta_S m_S + \theta_L m_L + \varphi)]a_{m_S m_L}. \tag{2.4}$$

Note that in general for cases with non-zero weights one needs the gauge transformation Ω^Φ to annihilate overall phase factors, in order for the order parameter to be invariant

Table 1. Continuous Abelian residual symmetry groups of superfluid ^{3}He.

	a_{11}	a_{10}	a_{1-1}	a_{01}	a_{00}
a_{11}	$(S-\Phi)\times(L-\Phi)$				
a_{10}	$S-\Phi$	$(S-\Phi)\times L$			
a_{1-1}	$S-\Phi$		$(S-\Phi)\times(L+\Phi)$		
a_{01}	$L-\Phi$	$J-\Phi$	$2S+L-\Phi$	$S\times(L-\Phi)$	
a_{00}		L	J	S	$S\times L$
a_{0-1}				S	
a_{-11}	$L-\Phi$	$S+2L-\Phi$	J		
a_{-10}		L			

under the rotations around the third axes. By inspection we now immediately write down table 1 which exhausts all possibilities for U(1) subgroups and products thereof. The notation is such that, for example, the residual symmetry group $U(1)^{2S+L-\Phi}$ denotes invariance of the state under transformations (2.4) with $\theta_S = 2\alpha$, $\theta_L = \alpha$ and $\varphi = -\alpha$. Phases with two residual U(1) factors are found along the diagonal and, hence, have only one non-zero entry (which may be taken equal to unity). For example, the so-called A_1 phase with residual symmetry $U(1)^{S-\Phi} \times U(1)^{L-\Phi}$ is characterised by $a_{11} \neq 0$. Phases with a single residual U(1) factor, on the other hand, have two or three non-zero entries. For example, the state with the residual symmetry group $U(1)^{2S+L-\Phi}$, which is the common U(1) factor in $U(1)^{S-\Phi} \times U(1)^{L+\Phi}$ ($a_{1-1} \neq 0$) and $U(1)^S \times U(1)^{L-\Phi}$ ($a_{01} \neq 0$), is described by a_{1-1}, $a_{01} \neq 0$. The so-called oblate or B_2 phase with residual symmetry

Table 2. Superfluid ^{3}He phases with a continuous Abelian residual symmetry. The parameters appearing in the matrices are non-zero; they are further arbitrary complex constants.

Residual symmetry group	$a_{m_S m_L}$	Name	Residual symmetry group	$a_{m_S m_L}$	Name
$SO(3)^J$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	B	$U(1)^{2S+L-\Phi}$	$\begin{pmatrix}0&0&u\\v&0&0\\0&0&0\end{pmatrix}$	
$U(1)^S$	$\begin{pmatrix}0&0&0\\u&v&w\\0&0&0\end{pmatrix}$		$U(1)^{S+2L-\Phi}$	$\begin{pmatrix}0&u&0\\0&0&0\\v&0&0\end{pmatrix}$	
$U(1)^{S-\Phi}$	$\begin{pmatrix}u&v&w\\0&0&0\\0&0&0\end{pmatrix}$		$U(1)^S \times U(1)^L$	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	Polar
$U(1)^L$	$\begin{pmatrix}0&u&0\\0&v&0\\0&w&0\end{pmatrix}$		$U(1)^S \times U(1)^{L-\Phi}$	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$	A
$U(1)^{L-\Phi}$	$\begin{pmatrix}u&0&0\\v&0&0\\w&0&0\end{pmatrix}$	A_2	$U(1)^{S-\Phi} \times U(1)^L$	$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$	β
$U(1)^J$	$\begin{pmatrix}0&0&u\\0&v&0\\w&0&0\end{pmatrix}$	B_2, oblate	$U(1)^{S-\Phi} \times U(1)^{L-\Phi}$	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	A_1
$U(1)^{J-\Phi}$	$\begin{pmatrix}0&u&0\\v&0&0\\0&0&0\end{pmatrix}$	ε			

group $U(1)^J$ has a_{1-1}, a_{00}, $a_{-11} \neq 0$. This group is the common U(1) factor in $U(1)^{S-\Phi} \times U(1)^{L+\Phi}$ ($a_{1-1} \neq 0$), $U(1)^S \times U(1)^L$ ($a_{00} \neq 0$) and $U(1)^{S+\Phi} \times U(1)^{L-\Phi}$ ($a_{-11} \neq 0$). Missing entries in table 1 correspond to symmetry groups that differ only in sign from the ones that are depicted, in the same way as $U(1)^{S-\Phi} \times U(1)^{L+\Phi}$ differs from $U(1)^{S-\Phi} \times U(1)^{L-\Phi}$ by one minus sign. In table 2 a summary is given. The states with only one coefficient (which is taken equal to one) are called inert states [6]. The other ones for which $a_{m_S m_L}$ explicitly depends on the physical parameters are called non-inert states. Note that upon interchanging the spin and orbital parts of a state the order parameter $a_{m_S m_L}$ is transformed into the transpose matrix with elements $a_{m_L m_S}$.

The symmetry breaking in the superfluid phases with non-zero weights as well as in the B phase is quite intricate. The non-triviality stems from the fact that the residual symmetries comprise transformations of different groups. As elucidated by Liu [7], the result of this is that the system is unable to distinguish between the different symmetry transformations which in turn give rise to some rather surprising physical phenomena [8, 9].

3. Discrete symmetries

We will now investigate the breaking down of the symmetry group (1.1) into discrete subgroups. For this we need the well known discrete subgroups of U(1) and SO(3) [10]. For U(1) they are given by the cyclic groups C_n in which the only symmetry consists of a single n-fold axis of symmetry. This group is of order n. For SO(3) the discrete subgroups are, besides C_n, the dihedral groups D_n of order $2n$, which have n twofold axes perpendicular to the principal C_n axis; the tetrahedral group T of order 12, the orthohedral group O of order 24, and the icosahedral group Y of order 60.

First, we explain that only the subgroups C_2^J, D_2^J, C_∞^J and D_∞^J may figure as residual symmetry groups of superfluid ^{3}He. Consider a common rotation around the third axis in spin and orbit space: $\Omega_3^S(\alpha)\Omega_3^L(\alpha)$,

$$a_{m_S m_L} \rightarrow a'_{m_S m_L} = e^{i\alpha m_J} a_{m_S m_L} \tag{3.1}$$

where $m_J \equiv m_S + m_L = 0, \pm 1, \pm 2$. A state is invariant under these rotations if

$$\alpha m_J = 0 \ (\mathrm{mod}\ 2\pi). \tag{3.2}$$

This condition restricts the allowed values of the rotation angle α when $m_J \neq 0$. (For the state with $m_J = 0$, i.e. a_{1-1}, a_{00}, $a_{-11} \neq 0$, α can take any value $0 \leq \alpha < 2\pi$. In fact, this is the oblate state we obtained in § 2.) The smallest (positive) value we find here is $\alpha = \pi$. The rotation over π need not be around the third axis, but may equally well be around the first axis: $\Omega_1^S(\pi)\Omega_1^L(\pi)$,

$$a_{m_S m_L} \rightarrow a'_{m_S m_L} = a_{-m_S - m_L} \tag{3.3}$$

or around the second one: $\Omega_2^S(\pi)\Omega_2^L(\pi)$,

$$a_{m_S m_L} \rightarrow a'_{m_S m_L} = (-1)^{m_J} a_{-m_S - m_L}. \tag{3.4}$$

As can easily be checked, equation (3.1) has only two solutions, namely C_∞^J and C_2^J. Also the set (3.1), (3.3) and (3.4) yields only two solutions, namely D_∞^J and D_2^J. In this way we arrive at table 3.

Most of the other states with discrete unbroken symmetry involve so-called colour groups [11] which are the discrete analogues of the continuous subgroups corresponding

Table 3. Largest subgroups of SO(3) in the $\underline{3}$ and $\underline{5}$ representations.

SO(3)	D_∞	C_∞	Y	O	T	D_4	D_3	D_2	C_4	C_3	C_2
$\underline{3}$		*									
$\underline{5}$	*							*			*

to non-zero weights encountered in § 2. The value of the order parameter of a state belonging to this non-trivial class is not invariant under certain discrete spin or/and orbital transformations, but is multiplied by a phase factor. Like in the continuous case, these factors must be compensated for by elements of $U(1)^{\Phi}$. That is, the residual symmetry groups of this class contain combined elements made up of elements of subgroups of $SO(3)^{S,L,J}$, $U(1)^{2S+L}$ or $U(1)^{S+2L}$ and of $U(1)^{\Phi}$. The construction of these extended groups [12] closely parallels the construction of magnetic point groups [13]. It proceeds in finding all the point groups that have, besides the trivial group I, the group C_2^J, D_2^J or $C_\infty^{S,L,J}$ as invariant subgroup. In the case of superfluid ^{3}He only these invariant subgroups have to be considered. This follows from the observation that upon neglecting the compensating $U(1)^{\Phi}$ factors, one is dealing with a $SO(3)^S \times SO(3)^L$ symmetry group in the $(\underline{3}^S, \underline{3}^L)$ representation to which table 3 applies. Note that only the continuous group SO(3) has D_∞ as invariant subgroup. Next, we must find among the point groups thus obtained, those groups G' for which the $\underline{3}$ or $\underline{5}$ representation contains a non-trivial one-dimensional representation with the property that the group elements that are represented by 1 form the invariant subgroup F that is being considered. In that case the group elements that are not represented by 1 form the factor group G'/F. They may be multiplied by the elements of the $U(1)^{\Phi}$ subgroup, which is isomorphic to the factor group, such that the combined elements are again represented by 1. In this way, the one-dimensional representation becomes the singlet representation, and hence the corresponding order parameter is invariant under the extended group. These colour groups are denoted by $G'(F)$.

We now list all possible colour groups that may be realised in superfluid ^{3}He as discrete residual symmetry groups. Observe that since we now have the possibility of compensating for overall phase factors, the smallest value of α (equation (3.1)) is half the value $\alpha = \pi$ that we found when no compensating factors were present. Consequently, the groups C_n, D_n with $5 \leq n < \infty$, and Y need not be considered. Those phases that may be obtained from others by simply interchanging the spin and orbital parts or by altering the signs of the weights m_S and m_L are omitted for brevity. We adopt the notation of [13] throughout the paper.

We start with the group C_2. The branching rule is

$$SO(3) \supset C_2 \qquad \underline{3} = A + 2B \qquad \underline{5} = 3A + 2B. \tag{3.5}$$

We recognise from the character table [13] that in the B representation, which is contained in both the $\underline{3}$ and the $\underline{5}$ representation, when the element c_2 is multiplied by the phase factor z_2 from the subgroup C_2^{Φ} of $U(1)^{\Phi}$, $z_n := \exp(i2\pi/n)$, the combined element is represented by the identity. Thus we find the extended groups $C_2^S(I)$, $C_2^S(I) \times U(1)^L$, $C_2^S(I) \times U(1)^{L-\Phi}$, and $C_2^J(I)$ which is the symmetry group of the so-called axiplanar phase or δ-phase. There is one more residual symmetry group based on C_2, namely the direct product $C_2^S(I) \times C_2^L(I)$.

For the group C_3 the branching rule is:

$$SO(3) \supset C_3 \qquad \underline{3} = A + E \qquad \underline{5} = A + 2E. \tag{3.6}$$

In the E representation of this group one must compensate with factors z_3 from the subgroup C_3^{Φ} of $U(1)^{\Phi}$. We find the groups $C_3^S(I)$ and $C_3^J(I)$. However, the order parameter belonging to the residual symmetry group $C_3^S(I)$ turns out to be the same as for $U(1)^{S-\Phi}$ of which it is a subgroup. Consequently, this subgroup does not lead to a new state. It is to be remarked that the group $C_3^{S+2L}(I)$, which can also be constructed here, may alternatively be denoted by $C_3^{S-L}(I)$, and belongs to the class we omit for brevity.

Next, we consider the group C_4:

$$\mathrm{SO}(3) \supset \mathrm{C}_4 \qquad \underline{3} = \mathrm{A} + \mathrm{E} \qquad \underline{5} = \mathrm{A} + 2\mathrm{B} + \mathrm{E}. \tag{3.7}$$

The E representation gives rise to the extended groups $C_4^S(I)$, $C_4^J(I)$ and $C_4^{S+2L}(I)$. The first two do not lead to new states as the order parameters are the same as for the continuous residual symmetry groups $U(1)^{S-\Phi}$ and $U(1)^{J-\Phi}$, respectively. From the B representation one obtains the group $C_4^J(C_2^J)$.

The branching rule for the group D_2 is

$$\mathrm{SO}(3) \supset \mathrm{D}_2 \qquad \underline{3} = \mathrm{B}_1 + \mathrm{B}_2 + \mathrm{B}_3 \qquad \underline{5} = 2\mathrm{A}_1 + \mathrm{B}_1 + \mathrm{B}_2 + \mathrm{B}_3. \tag{3.8}$$

Here, we only find the residual symmetry group $D_2^J(C_2^J)$. This colour group may be based on either the B_1, the B_2 or the B_3 representation. The corresponding order parameters differ slightly in form, and will be labelled 1, 2 or 3. Breaking down to the group $D_2^S(C_2^S)$ cannot be realised in superfluid ^{3}He. The reason is, as we explained earlier in this section, that if we neglect the compensating $U(1)^{\Phi}$ factors we have a $SO(3)^S \times SO(3)^L$ theory, and the extended group $D_2^S(C_2^S)$ collapses to C_2^S. But in the $(\underline{3}^S, \underline{3}^L)$ representation C_2^S is not a largest subgroup to which the symmetry can be broken down.

The A_2 representation of the group D_3

$$\mathrm{SO}(3) \supset \mathrm{D}_3 \qquad \underline{3} = \mathrm{A}_2 + \mathrm{E} \qquad \underline{5} = \mathrm{A}_1 + 2\mathrm{E} \tag{3.9}$$

gives rise to the colour group $D_3^J(C_3^J)$. The corresponding order parameter, however, describes a state with the larger residual symmetry $D_\infty^J(C_\infty^J)$. This state is called the planar or 2D phase.

For the group D_4 the branching rule is

$$\mathrm{SO}(3) \supset \mathrm{D}_4 \qquad \underline{3} = \mathrm{A}_2 + \mathrm{E} \qquad \underline{5} = \mathrm{A}_1 + \mathrm{B}_1 + \mathrm{B}_2 + \mathrm{E}. \tag{3.10}$$

Only the colour group $D_4^J(D_2^J)$, which may be constructed from the B_1 or the B_2 representation of D_4, leads to new states. The two order parameters have a sign difference and will be labelled 1 or 2.

Next, we consider the group T. The branching rule is

$$\mathrm{SO}(3) \supset \mathrm{T} \qquad \underline{3} = \mathrm{T} \qquad \underline{5} = \mathrm{E} + \mathrm{T}. \tag{3.11}$$

We find that breaking down to the extended group $T^J(D_2^J)$, which is based on the E representation, might happen in superfluid ^{3}He. The corresponding phase is called the α-phase. Finally, none of the representations contained in the $\underline{3}$ and $\underline{5}$ representations of the group O is one-dimensional. Hence, no colour groups that may act as residual symmetry groups in superfluid ^{3}He can be constructed from O. This completes the list of superfluid phases in which the symmetry group is broken down to a generalised magnetic point group where elements of subgroups of $U(1)^{\Phi}$ figure as compensating factors. We have thus obtained all previously found superfluid ^{3}He phases except the

Table 4. Superfluid ^{3}He phases with a discrete residual symmetry group. The coefficients appearing in the matrices are non-zero; they are further arbitrary complex constants.

Residual symmetry group	$a_{m_S m_L}$	Name	Residual symmetry group	$a_{m_S m_L}$	Name
$C_2^S(I)$	$\begin{pmatrix} u & v & w \\ 0 & 0 & 0 \\ x & y & z \end{pmatrix}$		D_2^J	$\begin{pmatrix} u & 0 & v \\ 0 & w & 0 \\ v & 0 & u \end{pmatrix}$	
$C_2^S(I) \times U(1)^L$	$\begin{pmatrix} 0 & v & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix}$		D_∞^J	$\begin{pmatrix} 0 & 0 & v \\ 0 & w & 0 \\ v & 0 & 0 \end{pmatrix}$	
$C_2^S(I) \times C_2^L(I)$	$\begin{pmatrix} u & 0 & w \\ 0 & 0 & 0 \\ x & 0 & z \end{pmatrix}$		$D_2^J(C_2^J)$	$\left\{ \begin{array}{l} \begin{pmatrix} u & 0 & v \\ 0 & 0 & 0 \\ -v & 0 & -u \end{pmatrix}_1 \\ \begin{pmatrix} 0 & u & 0 \\ v & 0 & -v \\ 0 & -u & 0 \end{pmatrix}_2 \\ \begin{pmatrix} 0 & u & 0 \\ v & 0 & v \\ 0 & u & 0 \end{pmatrix}_3 \end{array} \right.$	
$C_2^S(I) \times U(1)^{L-\Phi}$	$\begin{pmatrix} u & 0 & 0 \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}$				
C_2^J	$\begin{pmatrix} u & 0 & v \\ 0 & w & 0 \\ x & 0 & y \end{pmatrix}$				
$C_2^J(I)$	$\begin{pmatrix} 0 & u & 0 \\ v & 0 & w \\ 0 & x & 0 \end{pmatrix}$	Axiplanar, δ	$D_4^J(D_2^J)$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}_{1,2}$	
$C_3^J(I)$	$\begin{pmatrix} 0 & u & 0 \\ v & 0 & 0 \\ 0 & 0 & w \end{pmatrix}$		$D_\infty^J(C_\infty^J)$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	Planar, 2D
$C_4^{S+2L}(I)$	$\begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ v & 0 & w \end{pmatrix}$		$T^J(D_2^J)$	$\begin{pmatrix} i\sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & i\sqrt{3} \end{pmatrix}$	α
$C_4^J(C_2^J)$	$\begin{pmatrix} u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix}$				

bipolar phase [6]. The symmetry group of this phase involves a generalisation of the colour groups [14] which we will not discuss.

Given the residual symmetry groups, the explicit form of the order parameter in the different states may easily be constructed using (2.4), (3.3) and (3.4) where in the right-hand sides of the last two equations additional $U(1)^\Phi$ phase factors may be included. Table 4 summarises our analysis. We see three inert states appearing here, namely the planar, the α- and the $D_4^J(D_2^J)$ phases.

4. Superfluid core states of ³He-B vortices

In this section we study the ordered phases that exist in the core of superfluid ^{3}He-B vortices [4]. The core of these vortices is so large that it may be considered as a macroscopic system that may undergo phase transitions. In a classic superfluid like He II, the vortex core of the normal liquid must always exist. This may be understood as follows. The superfluid momentum $\boldsymbol{p}_s$ in a classic superfluid is given by

$$\boldsymbol{p}_s = \nabla\varphi \tag{4.1}$$

where φ is the $U(1)^\Phi$ Goldstone mode and, thus, the superflow is a potential flow. For a vortex with non-zero winding number n, i.e.

$$\frac{1}{2\pi}\oint \nabla\varphi \cdot \mathrm{d}\boldsymbol{l} \equiv n \neq 0 \tag{4.2}$$

where the line integral is taken over one circuit around the vortex, φ changes by a fixed amount around the contour. Consequently, if one shrinks this contour down to zero area one eventually encounters a region in which the field $\nabla\varphi$, and hence the kinetic energy, would diverge. In order to avoid this real-space singularity in the vortex core the $U(1)^\Phi$ symmetry is restored, and the liquid is normal there. (Note that the potential-flow character of the superfluid implies that the vorticity is concentrated in the vortex core.) In contrast, ^{3}He-B possesses vortices that display no destruction of superfluidity in their cores. This comes about because of the non-trivial internal structure of the superfluid ^{3}He phases. As a result the simple argument just given for the classic superfluid is not always valid in that the superfluid momentum can have a non-potential character. Moreover, some superfluid phases have quantum numbers that are compatible with those of certain vortices in ^{3}He-B. As first observed by Volovik and Mineev [15], vortices with a superfluid core can escape the vortex singularity in real space by transforming it into momentum space. Consequently, the order parameter vanishes only for certain momenta rather than for all momenta, as we have seen to be the case with real-space singularities. This observation also illustrates that the internal structure of a vortex-core phase must be non-trivial, since it has to exhibit these momentum-space singularities. In addition, it shows that real-space topology becomes coupled with the topology of the momentum space [4]. The theoretical methods employed up to now to study these vortices are based on numerically solving the Ginzburg–Landau equations. We will take a different route, one that is based on representation theory and that fits well into the scheme we developed earlier in this paper.

The key point in understanding which superfluid phases may figure as inner vortex-core phases is the quantum numbers of the state. In ^{3}He-B, as in a classical superfluid, the winding number of an axisymmetric vortex coincides with the total angular momentum quantum number of the vortex [4]. Hence, the topological charge is a good quantum number for these vortices and the quantum number m_J of the inner vortex-core phase must have this value. From this observation we infer that axisymmetric vortices with winding number $n \geqslant 3$ always have a normal core, since there is no superfluid ^{3}He phase that has a value $m_J \geqslant 3$ (see § 2). Furthermore, by recalling that the state with $m_J = 2$ is the A_1 phase, we conclude that this state may figure as the inner vortex-core state of a doubly quantised axisymmetric vortex ($n = 2$). It also follows that the superfluid inner core of a singly quantised axisymmetric vortex ($n = 1$) consists of the ε-phase ($m_J = 1$). Salomaa and Volovik [4] in their numerical analysis found this same state, but they erroneously identified it as a mixture of the β-phase and A phase. This is due to the facts that (i) the order parameter of the ε-phase is a linear combination of the order parameters of the β-phase and A phase:

$$a^{(\varepsilon)}_{m_S m_L} = r a^{(\beta)}_{m_S m_L} + s a^{(\mathrm{A})}_{m_S m_L} \tag{4.3}$$

where r and s are complex parameters, and (ii) the momentum-space topology of the A phase is identical to that of the ε-phase. Finally, it is amusing to note that axisymmetric vortices belonging to the trivial class ($n = 0$) may possess a superfluid ^{3}He-B$_2$ core, since this state has quantum number $m_J = 0$.

According to Salomaa and Volovik [4] there is experimental evidence for the existence of two different vortex-core states in superfluid ^{3}He-B separated by a first-order phase transition. As a mechanism for the vortex-core transition they suggested the breaking of axisymmetry [4]. They surmise that the first-order nature of the transition follows from the change in the (real-space) topology between the two vortex cores. The two vortices involved are the singly quantised axisymmetric and C$_2$-symmetric v-vortex [4]. The former belongs to the class with the ε inner vortex-core phase we discussed above. In the latter the axisymmetry is broken down to C_2^J, i.e. only a rotation by the angle π about the vortex axis is still a symmetry operation. Unlike the axisymmetric one, this C$_2$-symmetric vortex cannot possess a superfluid ε inner core, as this phase has the full $U(1)^J$ symmetry (up to phase transformations). Rather, it will possess a superfluid axiplanar core. In this state the symmetry group $U(1)^{J-\Phi}$ of the ε-phase is broken down to the subgroup $C_2^J(I)$. Thus the symmetry of the inner core state, C_2^J (up to a phase transformation), is again compatible with the symmetry of the vortex state. This non-axisymmetric solution was first obtained numerically by Thuneberg [16].

The singly and doubly quantised vortices will have a superfluid core, unless additional discrete symmetries of the vortices are incompatible with the symmetry of the core state. This is the case for singly quantised axisymmetric vortices with space parity (P) symmetry. As shown in [4], if the P operation is a symmetry operation the vortex solutions split into two disjoint classes, one with m_J even and the other with m_J odd. From this observation we infer that the singly quantised axisymmetric vortex with P symmetry cannot have a superfluid ε-core, since this phase and superfluid ^{3}He-B belong to a different class. For the doubly quantised vortices and the vortices belonging to the trivial class, P symmetry causes no problem, because the A$_1$ and B$_2$ phases belong to the same class as ^{3}He-B. We hope to return to these points in a later paper.

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